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The mathematical structure of incomplete constructive calculus

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Introduction. Errett Bishop developed "constructive analysis" in order to realize his mathematical philosophy--that is, only concrete constructions of mathematical objects from rationals can be trusted.

I was attracted to the mathematical structure of the outcomes of Bishop's constructions rather than the constructions themselves. Furthermore, having noticed the fact that Cauchy sequences (of reals) converge in his world due to a specific requirement imposed upon reals, I thought it more natural not to expect the convergence of Cauchy sequences, or any exact limit properties, in the more general setting. For these reasons, we take the algebraic properties of constructive reals as the axioms and develop constructive calculus within a formal system of intuitionistic logic. We will then notice that the constructive calculus is "interval-oriented" rather than "point-based," (This is so with or without convergence of Cauchy sequences.) and so we propose to review the constructive calculus in terms of rational intervals (open intervals with rational endpoints).

Section 1. The formal structure of (incomplete) constructive calculus

The reader may understand the content of this section as the formal development of Bishop's real calculus without assuming the convergence of Cauchy sequences.

First we set up the syntax.

Definition 1.1. Syntax

1) Sorts: N,Q,R

These are intended to represent respectively the set of natural numbers, the set of rationals and that of reals.

2) Types: N,Q,R; if $t_1,t_2,\dots,t_n,t$ are types, then so is $(t_1,t_2,\dots,t_n)\text{-->}t$.

Types represent set theoretical complexities of mathematical objects.

3) Variables of each type: in particular, l,m,n,... are of sort N, p,q,r,ε, δ,... are of Q, and a,b,c,x,y,z,... are of R.

4) Constants: 0,1,+,$\cdot$,=,<,$\leq$,max,min,-,$\div$,| |,symbols for primitive recursive functions, etc.

5) Term-formations: Terms are defined from variables and constants by applications of function symbols. A term is a formal expression of a mathematical object such as $(a+1)\cdot(1+1)$, $\max\{|a|,b-1\}$,....

6) Logical connectives: $\neg$(not), &(and), v(or), |-(implies), $\forall$(for all), $\exists$(there exists). --> may also be used for implication.

7) Formula-formations: Formulas are constructed from atomic relations of terms by applications of logical connectives. (A formula is the formal expression of a mathematical assertion such as $\forall x \exists y(x+(1+1+1)=y)$.)

Definition 1.2. Mathematical axioms. Mathematical axioms of constructive calculus are basic relations of numbers and functions, which are consequences of Bishop's construction. We list a few of them.

1) Equality axioms on = for each sort.
2) Characterizations of N,Q,R.
3) The usual arithmetic of N and Q.
4) The arithmetic of R.

$x=z \,\&\, y\leq u \to x+y\leq z+u$

$\max\{x,y\}\geq x,y$

$|x|=\max\{x,-x\}$

$|x|>y \to x>y \vee -x>y$

Note. It is not claimed that $\max\{x,y\}=x$ or y; $x \# y$ (which abbreviates $x<y \vee x>y$) is not equivalent to $\neg x=y$.

$x \# 0 \to x\cdot x^{-1}=1$

$\forall x\forall y(x<y \vdash \exists p(x<p<y))$

5) $a=0 \to \forall n(|a|<1/n)$

Limit property(*) $\forall n(|a|<1/n) \to a=0$ (In fact we develop our theory without this.)

Note. The following do not necessarily hold; the law of trichotomy of reals; $x \leq y \rightarrow x<y \vee x=y$. It is thus noticed that the system of reals is not linearly ordered; it is a partially ordered structure.)

Definition 1.3. System.

The system $\mathcal{R}$ in which the constructive calculus is to be developed is based on three-sorted intuitionistic predicate calculus, where the quantifier-free first order part without the real-sort is classical. That is, the excluded middle (or the proof by contradiction) is not admitted except for quantifier-free properties of natural numbers and rationals (that is, decidable ones). The axioms are those above.

Note. The atomic relations such as = and < of sorts N and Q are decidable(computable), and hence the excluded middle is admitted for these.

Corollary. $y<z \rightarrow x<z \vee x>y$.

Definition 1.4. A sequene of reals, $\{x_n\}$, is "convergent":

$$\forall \varepsilon>0 \exists r_1, r_2 (0<r_2-r_1<\varepsilon \,\&\, \exists m \forall n=m (r_1<x_n<r_2)).$$

That is, $\{x_n\}$ does not necessarily converge to an admitted real, but the location of convergence can be pinpointed as accurately as desired.

In the following, the propositions are those of our system $\mathcal{R}$.

Proposition 1.1. $\{x_n\}$ is convergent if and only if it is Cauchy.

Definition 1.5. "The lub of A, a set of reals, is definable":

$$\forall \varepsilon>0 \exists r_1, r_2 (0<r_2-r_1<\varepsilon \,\&\, \exists a \in A(r_1<a) \,\&\, \forall a \in A(a<r_2))$$

f is continuous on a bounded, closed interval I:

$$\forall \varepsilon>0 \exists \delta>0 \forall x,y \in I(|x-y| \leq \delta \vdash |f(x)-f(y)| \leq \varepsilon)$$

f is continuous on an interval J: f is continuous on any subinterval I which is bounded and closed.

Proposition 1.2. If f is continuous on I, bounded and closed, then lub off(I) and glb of f(I) are "definable."

Proposition 1.3(Intermediate value theorem). Suppose f is continuous on I=[a,b], where f(a)<0<f(b). Then,

$$\forall \varepsilon>0\exists(s_1,s_2)\subset I(f((s_1,s_2))\subset(-\varepsilon,\varepsilon)).$$

Outline of the proof. Notice that $|f(x)|$ is continuous on I. Put A=|f|(I). glb of A is "definable." So $\forall\varepsilon>0\exists(r_1,r_2)$. Suppose $r_1>0$. Then $f(a)<-r_1<0$ and $f(b)>r_1>0$. Let $\delta>0$ be a modulus of continuity of f for r_1. We can choose $x_0,x_1,\ldots,x_n$, so that $a=x_0\leq x_1\leq\ldots\leq x_n=b$ and $0\leq x_{k+1}-x_k<\delta$. Then we can deduce in our reasoning that $f(x_k)<0$ for all $k\leq n$, and hence f(b)<0,resulting in a Contradiction. That is, $r_1>0$ yields a contradiction. Since the order relation of sort Q is decidable, this implies $r_1\leq 0$, which leads to $r_2<\varepsilon$. So, $z\in A\&z<r_2\dashrightarrow z\in A\&z<\varepsilon$, and hence $\exists x\in I(|f(x)|<\varepsilon)$. By virtue of the continuity of f, we can find an $x\in(s_1,s_2)\subset I$, so that $f((s_1,s_2))\subset(-\varepsilon,\varepsilon)$. Notice that we have shown the proposition without the completeness of reals.

Definition 1.6. Let f and g be continuous on I, bounded and closed. g is a derivative of f on I:

$$\forall\varepsilon>0\exists\delta>0\forall x,y\in I(|x-y|\leq\delta\dashrightarrow|f(y)-f(x)-g(x)(y-x)|\leq\varepsilon|y-x|)$$

g is unique if existent, which we write as f'.

Note. In the definitions of continuity and derivatives, δ, the modulus of continuity or differentiability,is claimed to exist individually to each $\varepsilon>0$, while in Bishop's case δ has to be given a priori as a computable function of ε. The reason for the sufficiency of our definitions is that our system $\mathcal{R}$ forces δ to be found concretely for ε.

Prososition 1.4(Rolle's theorem).

$$\forall\varepsilon>0\exists(r_1,r_2)f'((r_1,r_2))\subset(-\varepsilon,\varepsilon).$$

Definition 1.7. Let f and F be continuous on I. We present two alternatives for indefinite integrals.

1) $F(x) = \int_a^x f(t)dt$:

$$\forall x \in I \forall \varepsilon>0 \exists N \forall n \geq N |F(x)-S(f,a,x,n)| < \varepsilon.$$

2) $F'(x) = f(x)$ on I.

We have not decided which should be employed.

Section 2. Neighborhood representation of reals -- elimination of reals from the calculus

We shall thus subsequently propose a development of "neighborhood calculus" within the framework of intuitionistic logic. This time sort R is irrelevant. We will never have to talk about reals (except for heuristic reasons). The idea is that if, for a real number a, the system of its basic neighborhoods with rational endpoints, say. $N(a)=\{(r_1,r_2);r_1<a<r_2\}$,is associated,then any $(r_1,r_2)\in N(a)$ will be regarded as an approximation to a.

In the following, intervals (r_1,r_2) are of rational endpoints, r_1, r_2; in fact it can be read through with rationals of finite decimals. We can consider an interval (r_1,r_2) to be a pair of r_1 and r_2 with the restriction that $r_1 \leq r_2$. In particular, (r,r) will be identified with $(0,0)$ and denoted as ϕ (the empty interval).

Constructive theory of natural numbers and rational numbers will be assumed.

Definition 2.1. Let $A=(r_1,r_2)$ and $B=(s_1,s_2)$.

1) $A\subset B$: $s_1 \leq r_1 \leq r_2 \leq s_2$

2) Arithmetic of intervals

$$A+B=(r_1+s_1,r_2+s_2), \quad -B=(-s_2,-s_1)$$

$A\cdot B=(\min\{r_1 s_1,r_1 s_2,r_2 s_1,r_2 s_2\}$, $1/B=(1/s_2,1/s_1)$, where if $s_1=0$, then $1/s_1=\infty$; if $s_2=0$, then; $1/s_2=-\infty$; if $0\in B$, then $1/B$ is undefined.

$\max(A,B) = t_1,t_2)$, where $t_2=\max(r_2,s_2)$, $t_1=r_1$ if $t_2=r_2>s_2$, $t_1=s_1$ if $t_2=s_2>r_2$ and $t_1=\max(r_1,s_1)$ if $r_2=s_2$.

$|A|=\max(A,-A)$

3) $A=B$: $r_1=s_1$ anad $r_2=s_2$

We have several alternatives of (partial) order of intervals.

$A <\!\cdot\, B$: $r_2 \leq s_1$ (presuming that A and B are not empty)

$A \lesssim B$: $r_2 \leq s_2$

$A<B$: $r_1 < s_1$ & $r_2 < s_2$

Corollary. 1) Arithmetics of intervals are (commutative and) associative. Other usual relations also hold except for order relations.

2) $A \lesssim B \to A+C \lesssim B+C$; $A < B \to A+C < B+C$

$A\phi=\phi$, $A+\phi=A$

3) $A<B \vee A=B \to A \lesssim B$, but not necessarily the converse.

4) $\max(A,B)=A$ or B.

Meta-proposition. 1) Let $*$ be one of $+$, $-$, $\cdot$, $\div$, and let a and b be reals. Then $a \in A$ & $b \in B \to a*b \in A*B$.

2) The operators and relations defined as above are all decidable; in fact primitive recursive.

3) A relation such as $A \cap B=\phi$ (or not) is decidable.

Definition 2.2. 1) Let $\mathcal{R}$ be the family of all rational intervals, and let $\mathcal{F}$ be any subfamily of $\mathcal{R}$. $\mathcal{F}$ is said to be inclusion closed if $B \subset A \in \mathcal{F}$ implies $B \in \mathcal{F}$.

2) A map F from $\mathcal{F}$ to $\mathcal{R}$ will be called a neighborhood transformation.

3) F is said to be (inclusion) monotone if $B \subset A \to F(B) \subset F(A)$.

Metadefinition. Let f be a real function. An (inclusion) monotone transformation F is called a realization of f if $f(A) \subset F(A)$ for every relevant A. In this case $f(c) \in \bigcap\{F(A); c \in A\}$.

Proposition 2.1. 1) $+$, $\cdot$, etc. are monotone.

2) If F is monotone, then $F(A \cap B) \subset F(A) \cap F(B)$.

Definition 2.3. 1) $\sigma(A) = r_2 - r_1$ if $A=(r_1, r_2)$ [the size of A].

2) A neighborhood transformation F is continuous if it is sum-preserving in the following sense: if $A \cup B$ is an interval, then $F(A \cup B)=F(A) \cup F(B)$, and furthermore the ε-δ relation holds. That is,

$$\forall \varepsilon>0\ \exists \delta>0\ \forall B(\sigma(B)<\delta \to \sigma(F(B))<\varepsilon)$$

3) A sequence of intervals $\{A_n\}$ is convergent (or Cauchy) if
$\exists l \forall n, m \geq l \quad A_n \cap A_m \neq \phi$.

4) Suppose $\mathcal{F} \subset \mathcal{N}$ is inclusion closed. lub $\mathcal{F}$ is definable by A if
$\forall B \in \mathcal{F} (B \lesssim A)$ and $\exists B \in \mathcal{F} (A \cap B \neq \phi)$

5) Suppose $U_1 < U_2$ in $\mathcal{N}$, where $\mathcal{N}$ contains all intervals between some two rationals. $[U_1, U_2] = \{U\} (U_1 \lesssim U \lesssim U_2)$.

6) Let F and G be continuous on $I=[U_1, U_2]$. G is a derivative of F if $0 \in B \rightarrow \exists C \ni 0 \forall U, V \in I (V-U \subset C \rightarrow (F(V)-F(U)-G(U)(V-U)) \subset B\cdot(V-U))$
A derivative of F will be denoted as F'.

Note. A sum-preserving transformation is monotone.

The neighborhood-version of some of the typical theorems of the calculus can be stated as follows.

Intermediate value theorem. Suppose F is continuous on $I=[U_1, U_2]$. Then

$$F(U_1) < 0 < F(U_2) \;\&\; \varepsilon > 0 \rightarrow \exists U \in I (F(U) \subset (-\varepsilon, \varepsilon))$$

Roll's theorem. Suppose F' is a derivative of F. Then

$$F(U_1) = F(U_2) \;\&\; 0 \in B \rightarrow \exists U \in I (F'(U) \cap B \neq \phi)$$

Remark. Our intended theory of neighborhood calculus can be formally developed in intuitionistic arithmetic, and hence various interpretations such as recursive realization can be applied. One can then extract desired objects in the theorems constructively.

The materical presented in this section is just a crude idea. We hope to develop it to a fruitful mathematical theory.

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